A TUTORIAL INTRODUCTION TO DIFFERENTIABLE MANIFOLDS AND CALCULUS ON MANIFOLDS

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In this tutorial I try by means of several examples to illustrate the basic definitions and concepts of differentiable manifolds. There are few proofs (not that there are ever many at this level of the theory). This material should be sufficient to understand the use made of these concepts in the other contributions in this volume, notably the lectures by Kliemann, and my own lectures on filtering; or at least, it should help in explaining the terminology employed. Quite generally in fact, it can be said that the global point of view, i.e. analysis on manifolds rather than on open pieces of $\mathbb{R}^n$, can have many advantages, also in areas like engineering where this approach is less traditional. This tutorial is a revised and greatly expanded version of an earlier one entitled 'A tutorial introduction to differentiable manifolds and vector fields' which appeared in M. HAZEWINKEL, J.C. WILLEMS (eds), Stochastic Systems: the mathematics of filtering and identification, Reidel, 1981, 77-93.

1. INTRODUCTION AND A FEW REMARKS

Roughly an n-dimensional differentiable manifold is a gadget which locally looks like $\mathbb{R}^n$, the space of all real vectors of length n, but globally perhaps not; A precise definition is given below in section 2. Examples are the sphere and the torus, which are both locally like $\mathbb{R}^2$ but differ globally from $\mathbb{R}^2$ and from each other.

Such objects often arise naturally when discussing problems in analysis (e.g. differential equations) and elsewhere in mathematics and its applications. A few advantages which may come about by doing analysis on manifolds rather than just on $\mathbb{R}^n$ are briefly discussed below.

1.1 Coordinate freeness ("Diffeomorphisms").

A differentiable manifold can be viewed as consisting of pieces of $\mathbb{R}^n$ which are glued together in a smooth (= differentiable) manner. And it is on the basis of such a picture that the analysis (e.g. the study of differential equations) often proceeds. This brings more than a mere extension of analysis on $\mathbb{R}^n$ to analysis on spheres, tori, projective spaces and the like; it stresses the "coordinate free approach", i.e. the formulation of problems and concepts in terms which are invariant under (non-linear) smooth coordinate transformations and thus also helped to bring about a better understanding even of analysis on $\mathbb{R}^n$. The more important results, concepts and definitions tend to be "coordinate free".

1.2 Analytic continuation.

A convergent power series in one complex variable is a rather simple object. It is considerably more difficult to obtain an understanding of the collection of all analytic continuations of a given power series, especially because analytic continuation along a full circuit (contour) may yield a different function value than the initial one. The fact that the various continuations fit together to form a Riemann surface (a certain kind of 2-dimensional manifold usually different from $\mathbb{R}^2$) was a major and most enlightening discovery which contributes a great deal to our understanding.

1.3 Submanifolds.
Consider an equation $\dot{x}=f(x)$ in $\mathbb{R}^n$. Then it often happens, especially in problems coming from mechanics, that the equation has the property that it evolves in such a way that certain quantities (e.g. energy, angular momentum) are conserved. Thus the equation really evolves on a subset $\{x\in\mathbb{R}^n : E(x)=c\}$ which is often a differentiable submanifold. Thus it easily could happen, for instance, that $\dot{x}=f(x)$, f smooth, is constrained to move on a (disorted) 2-sphere which then immediately tells us that there is an equilibrium point, i.e. a point where $f(x)=0$. This is the so-called hairy ball theorem which says that a vectorfield on a 2-sphere must have a zero; for vectorfields and such, cf below.

Also one might meet 2 seemingly different equations, say, one in $\mathbb{R}^4$ and one in $\mathbb{R}^3$ (perhaps both intended as a description of the same process) of which the first has two conserved quantities and the second one. It will then be important to decide whether the surfaces on which the equations evolve are diffeomorphic, i.e. the same after a suitable invertible transformation and whether the equations on these submanifolds correspond under these transformations.

1.4 Behaviour at infinity.
Consider a differential equation in the plane $\dot{x}=P(x,y)$, $\dot{y}=Q(x,y)$. To study the behaviour of the paths far out in the plane and such things as solutions escaping to infinity and coming back, Poincaré already completed the plane to real projective 2-space (an example of a differential manifold). Also the projective plane is by no means the only smooth manifold compactifying $\mathbb{R}^2$ and it will be of some importance for the behaviour of the equation near infinity whether the "right" compactification to which the equation can be extended will be a projective 2-space, a sphere, or a torus, or ..., or, whether no such compactification exists at all. A good example of a set of equations which are practically impossible to analyse completely without bringing in manifolds are the matrix Riccati equations which naturally live on Grassmann manifolds. The matrix Riccati equation is of great importance in linear Kalman-Bucy filtering. It also causes major numerical difficulties. It will therefore return below by way of example.

1.5 Avoiding confusion between different kinds of objects.
Consider an ordinary differential equation $\dot{x}=f(x)$ on $\mathbb{R}^n$, where $f(x)$ is a function $\mathbb{R}^n\to\mathbb{R}^n$. When one now tries to generalize this idea of a differential equation on a manifold one discovers that $\dot{x}$ and hence $f(x)$ is a different kind of object; it is not a function, but, as we shall see, it is a vectorfield; in other words under a nonlinear change of coordinates the right hand side of such a differential equation $\dot{x}=f(x)$ transforms not as a function, but in a different way (involving Jacobian matrices, as everyone knows).

2. Differentiable manifolds
Let U be an open subset of $\mathbb{R}^n$, e.g. an open ball. A function $f:U\to\mathbb{R}$ is said to be C^∞ or *smooth* if all partial derivatives (any order) exist at all $x\in U$. A mapping $\mathbb{R}^n\supset U\to\mathbb{R}^m$ is smooth if all components are smooth; $\phi:U\to V$, $U\subset\mathbb{R}^n$, $V\subset\mathbb{R}^n$ is called a diffeomorphism if ϕ is 1-1, onto, and both ϕ and ϕ^{-1} are smooth.

As indicated above a smooth n-dimensional manifold is a gadget consisting of open pieces of $\mathbb{R}^n$ smoothly glued together. This gives the following pictorial definition of a smooth n-dimensional manifold M (fig. 1).

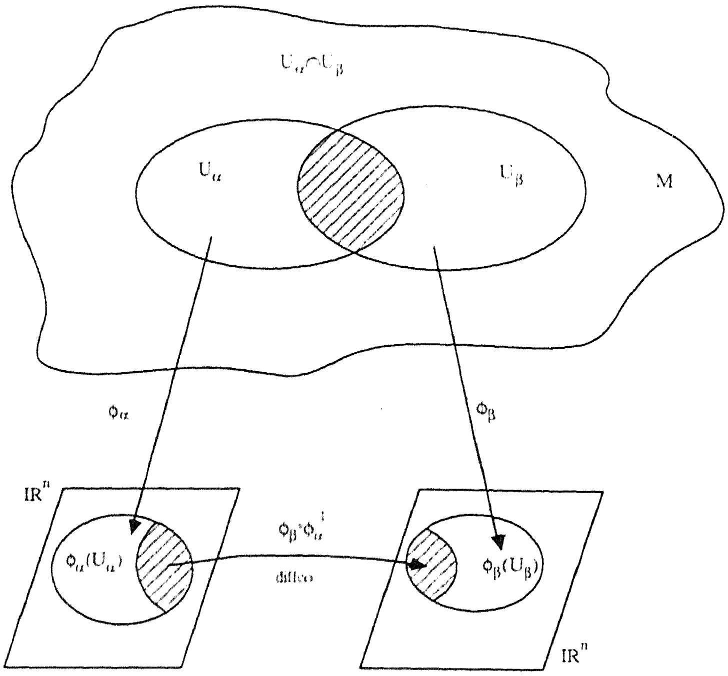

FIGURE 1. Pictorial definition of a differentiable manifold.

2.1 Example.
The circle $S^1 = \{(x_1, x_2): x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$

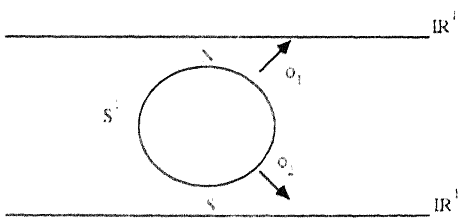

FIGURE 2. Example: the circle

$U_1 = S^1 \setminus \{S\}$, $U_2 = S^1 \setminus \{N\}$ so $U_1 \cup U_2 = S^1$. The "coordinate charts" ϕ_1 and ϕ_2 are given by

$$\phi_1(x_1,x_2) = \frac{x_1}{1+x_2}, \quad \phi_2(x_1,x_2) = \frac{x_1}{1-x_2}$$

Thus $\phi_1(U_1 \cap U_2) = \mathbb{R} \setminus \{0\}$, $\phi_2(U_1 \cap U_2) = \mathbb{R} \setminus \{0\}$ and the map $\phi_2 \circ \phi_1^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ is given by $x \mapsto x^{-1}$ which is a diffeomorphism.

2.2 Formal definition of a differentiable manifold.
The data are

- M, a Hausdorff topological space
- A covering $\{U_\alpha\}_{\alpha \in I}$ by open subsets of M
- Coordinate maps $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^n$, $\phi_\alpha(U_\alpha)$ open in $\mathbb{R}^n$.

These data are subject to the following condition

- $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism.

Often one also adds the requirement that M be paracompact. We shall however disregard these finer points; nor shall we need them in this volume.

2.3 Constructing differentiable manifolds 1: embedded manifolds.
Let M be a subset of $\mathbb{R}^N$. Suppose for every $x \in M$ there exists an open neighbourhood $U \subset \mathbb{R}^n$ and a smooth function $\psi : U \to \mathbb{R}^N$ mapping U homeomorphically onto an open neighbourhood V of x in M. Suppose moreover that the Jacobian matrix of ψ has rank n at all $u \in U$. Then M is a smooth manifold of dimension n. (Exercise; the coordinate neighbourhoods are the V's and the coordinate maps are the ψ^{-1}; use the implicit function theorem). Virtually the same arguments show that if $\phi : U \to \mathbb{R}^k$, $U \subset \mathbb{R}^{n+k}$, is a smooth map and the rank of the Jacobian matrix $J(f)(x)$ is k for all $x \in \phi^{-1}(0)$, then $\phi^{-1}(0)$ is a smooth n-dimensional manifold. We shall not pursue this approach but concentrate instead on:

2.4 Constructing differentiable manifolds 2: gluing.
Here the data are as follows

- an index set I
- for every $\alpha \in I$ an open subset $U_\alpha \subset \mathbb{R}^n$
- for every ordered pair (α,β) an open subset $U_{\alpha\beta} \subset U_\alpha$
- diffeomorphisms $\phi_{\alpha\beta} : U_{\alpha\beta} \to U_{\beta\alpha}$ for all $\alpha,\beta \in I$

These data are supposed to satisfy the following compatibility conditions

- $U_{\alpha\alpha} = U_\alpha$, $\phi_{\alpha\alpha} = id$
- $\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}$ (where appropriate)

(where the last identity is supposed to imply also that $\phi_{\alpha\beta}(U_{\alpha\beta} \cap U_{\beta\gamma}) \subset U_{\beta\gamma}$ so that $\phi_{\alpha\beta}(U_{\alpha\beta} \cap U_{\alpha\gamma}) = U_{\beta\gamma} \cap U_{\beta\alpha}$).

These are not yet all conditions, cf below, but the present lecturer, e.g., has often found it advantageous to stop right here so to speak, and to view a manifold simply as a collection of open subsets of $\mathbb{R}^n$ together with gluing data (coordinate transformation rules).

From the data given above one now defines an abstract topological space M by taking the disjoint union of the U_α and then identifying $x \in U_\alpha$ and $y \in U_\beta$ iff $x \in U_{\alpha\beta}$, $y \in U_{\beta\alpha}$, $\phi_{\alpha\beta}(x) = y$. This gives a natural injection $U_\alpha \to M$ with image U'_α say. Let $\phi_\alpha : U'_\alpha \to U_\alpha$ be the inverse map. The $\phi_\alpha : U'_\alpha \to U_\alpha \subset \mathbb{R}^n$ define local coordinates on M. Then this gives us a differentiable manifold M in the sense of definition 2.2 provided that M is Hausdorff and paracompact, and these are precisely the conditions which must be added to the gluing compatibility conditions above.

2.5 Functions on a "glued manifold".
Let M be a differentiable manifold obtained by the gluing process described in 2.4 above. Then a differentiable function $f:M \rightarrow \mathbb{R}$ consist simply of a collection of functions $f_\alpha : U_\alpha \rightarrow \mathbb{R}$ such that $f_\beta \circ \phi_{\alpha\beta} = f_\alpha$ on $U_{\alpha\beta}$, as illustrated in fig. 3.

Thus for example a function on the circle S^1, cf figure 2, can be described either as a function of two variables restricted to $S^1 \subset \mathbb{R}^2$ or as two functions f_1, f_2 of one variable on U_1 and U_2 such that $f_1(x) = f_2(x^{-1})$. Obviously the latter approach can have considerable advantages.

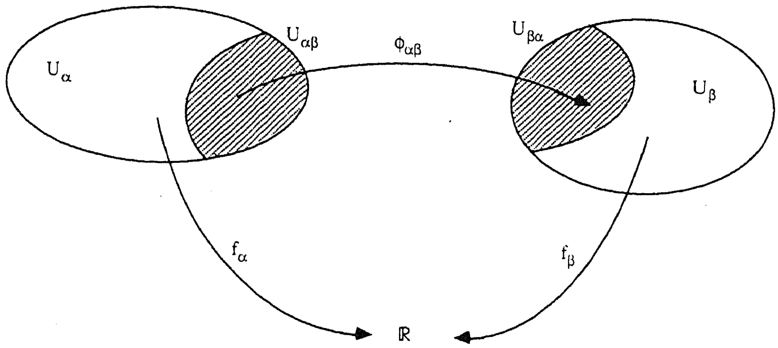

FIGURE 3. Functions on a glued manifold

2.6 Example of a 2 dimensional manifold: the Möbius band.
The (open) Möbius band is obtained by taking a strip in $\mathbb{R}^2$ as indicated below in fig. 4 without its upper and lower edges and identifying the left hand and right hand edges as indicated.

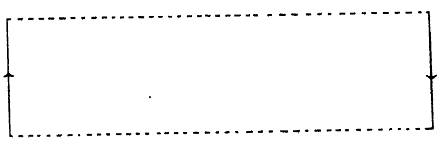

FIGURE 4. Construction of the Möbius band

The resulting manifold (as a submanifold of $\mathbb{R}^3$) looks something like the following figure 5.

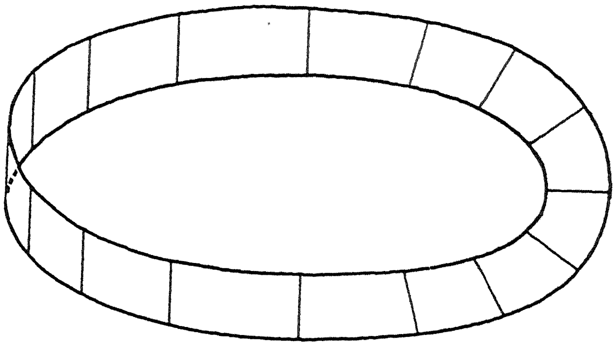

FIGURE 5. The Möbius band

It is left as an exercise to the reader to cast this description in the form required by the gluing description of 2.4 above. The following pictorial description (fig. 6) will suffice.

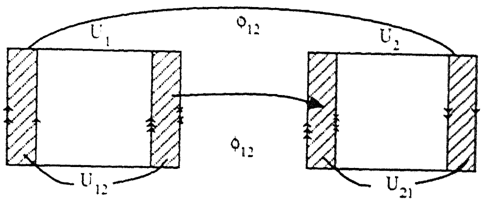

FIGURE 6. Gluing description of the Möbius band

2.7 Example: the 2-dimensional sphere.
The picture in fig. 7 below shows how the 2-sphere $S^2 = \{x_1, x_2, x_3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ can be obtained by gluing two disks together. If the surface of the earth is viewed as a model for S^2 (or vice versa, which is the more customary use of the world 'model'), the first disk covers everything north of Capricorn and the second everything south of Cancer.

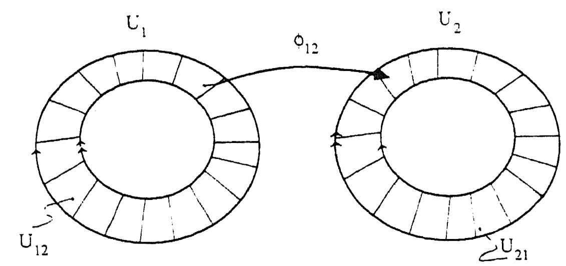

FIGURE 7. Gluing description of the 2-sphere S^2

2.8 Example. The Grassmann manifolds $Gr_k(\mathbb{R}^n)$.
As a set $Gr_k(\mathbb{R}^n)$ consists of all k-dimensional subspaces of $\mathbb{R}^n$. Thus $Gr_1(\mathbb{R}^n)$ is real projective space of dimension $n-1$ and in particular $Gr_1(\mathbb{R}^2)$ is the real projective line, i.e. the circle. We shall now also present a gluing data description of $Gr_k(\mathbb{R}^n)$. To this end it is useful to introduce the following notation. Let A be an $k \times n$ matrix, $k<n$ and let α be a subset of $\{1, \ldots, n\}$ of size k. Then A_α denotes the $k \times k$ matrix obtained from A by removing all columns whose index is not in α.

Now let U_α be the set of all $k \times n$ matrices A such that $A_\alpha = I_k$, the $k \times k$ identity matrix

$$U_\alpha = \{A \in \mathbb{R}^{k \times n} : A_\alpha = I_k\}$$

Because the entries a_{ij} with $j \in \{1, \ldots, n\} \setminus \alpha$ of these matrices are arbitrary this is clearly just a slightly crazy way of writing down all real $k \times (n-k)$ matrices or, in other words, all real $k(n-k)$ vectors, i.e. $U_\alpha \simeq \mathbb{R}^{k(n-k)}$.
The gluing data for $Gr_k(\mathbb{R}^n)$ are now as follows
- the index set I consists of all subsets α of size k of $\{1, \ldots, n\}$
- for each α, $U_\alpha = \mathbb{R}^{k(n-k)}$ realized as indicated above
- for each ordered pair of indices α, β

$$U_{\alpha\beta} = \{A \in U_\alpha : A_\beta \text{ is invertible}\}$$

- the diffeomorphsims

$$\phi_{\alpha\beta} : U_{\alpha\beta} \to U_{\beta\alpha}$$

are given by

$$A \mapsto (A_\beta)^{-1} A$$

We shall see below (in 2.12) that $Gr_k(\mathbb{R}^n)$ is indeed the space of all k-dimensional subspaces of $\mathbb{R}^n$.

2.9 Exercise.
Check that the compatibility conditions $\phi_{\alpha\alpha} = id$ and $\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}$ of 2.4 above hold. Prove also that the manifold obtained from these gluing data is Hausdorff.

2.10 Morphism of differentiable manifolds.
Let M and N be differentiable manifolds obtained by the gluing process of section 2.4 above. Say M is obtained by gluing together open subsets U_α of $\mathbb{R}^n$ and N by gluing together open subsets V_β of $\mathbb{R}^m$. Then a smooth map $f:M \to N$ (a morphism) is given by specifying for all α,β an open subset $U_{\alpha\beta} \subset U_\alpha$ and a smooth map $f_{\alpha\beta}:U_{\alpha\beta} \to V_\beta$ such that $\bigcup_\beta U_{\alpha\beta} = U_\alpha$ and the $f_{\alpha\beta}$ are compatible under the identifications $\phi_{\alpha\alpha'}:U_{\alpha\alpha'} \to U_{\alpha'\alpha}, \phi_{\beta\beta'}:V_{\beta\beta'} \to V_{\beta'\beta}$, i.e. $f_{\alpha'\beta'} \circ \phi_{\alpha\alpha'} = \phi_{\beta\beta'} \circ f_{\alpha\beta}$ whenever appropriate. (Here the ϕ'_s are the gluing diffeomorphisms for M and the ψ'_s are the gluing diffeomorphisms for N).

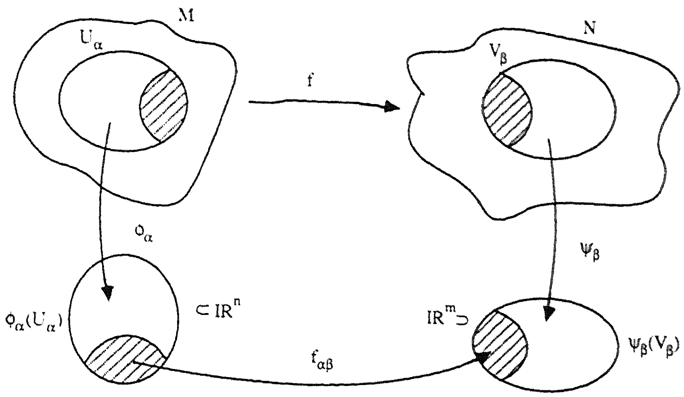

FIGURE 8. Morphisms

2.11 Exercise:
Show that the description of the circle S^1 as in 2.1 above gives an injective morphism $S^1 \to \mathbb{R}^2$.

2.12 Example: Grassmann manifolds continued.
Let $\mathbb{R}^{k\times n}_{reg}$ be the open subset of $\mathbb{R}^{k\times n} = \mathbb{R}^{kn}$ consisting of all $k\times n$ matrices of maximal rank k. (Recall that $k < n$.) We are going to define a differentiable morphism

$$\pi: \mathbb{R}^{k\times n}_{reg} \to Gr_k(\mathbb{R}^n)$$

by the method of section 2.10 above. In this case $\mathbb{R}^{k\times n}_{reg} = U \subset \mathbb{R}^{kn}$ is defined by a single open subset. Thus we need for each α an open subset V_α of U and a smooth map $\pi_\alpha:V_\alpha \to U_\alpha$ where U_α is as above in 2.8. These data are defined as follows

$$V_\alpha = \{M \in \mathbb{R}^{k\times n}_{reg} : M_\alpha \text{ is invertible}\}$$

$$\pi_\alpha : V_\alpha \to U_\alpha,\quad M \mapsto (M_\alpha)^{-1} M \in U_\alpha$$

It is an easy exercise (practically identical with the first part of exercise 2.9) to check that the required compatibility conditions are met.

It is now simple to see that $Gr_k(\mathbb{R}^n)$ as defined in 2.8 is indeed the space of all k-dimensional subspaces of $\mathbb{R}^n$. Indeed let W be such a subspace. Choose a basis for $W \subset \mathbb{R}^n$. These k n-vectors written as row vectors define $k\times n$ matrix $A(W)$ in $\mathbb{R}^{k\times n}_{reg}$. Taking a different basis for W amounts to replacing $A(W)$ with $SA(W)$ where S is an invertible $k \times k$ matrix. Now

$$(SA(W))_\alpha = S(A(W))_\alpha$$

and it follows that if $A(W) \in V_\alpha$ then also $SA(W) \in V_\alpha$ and that moreover

$$\pi(SA(W)) = \pi A(W)$$

Thus every k-dimensional vectorspace in $\mathbb{R}^n$ defines a unique point of $Gr_k(\mathbb{R}^n)$ and vice versa. ($A \in U_\alpha$ is of maximal rank and hence defines a k dimensional vectorspace.)

3. Differentiable vectorbundles

Intuitively a vectorbundle over a space S is a family of vectorspaces parametrized by S. Thus for example the Möbius band of example 2.6 can be viewed as a family of open intervals in $\mathbb{R}$ parametrized by the circle, cf fig. 9 below, and if we are willing to identify the open intervals with $\mathbb{R}$ this gives us a family of one dimensional vectorspaces parametrized by S^1 which locally (i.e. over small neighbourhoods in the base space S^1) looks like a product but globally is not equal to a product.

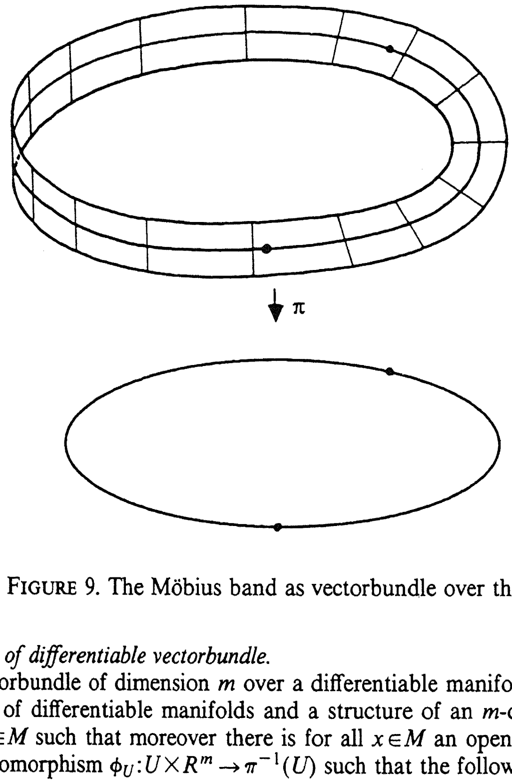

Figure 9. The Möbius band as vectorbundle over the circle

3.1 Formal definition of differentiable vectorbundle.

A differentiable vectorbundle of dimension m over a differentiable manifold M consists of a surjective morphism $\pi : E \rightarrow M$ of differentiable manifolds and a structure of an m-dimensional real vectorspace on $\pi^{-1}(x)$ for all $x \in M$ such that moreover there is for all $x \in M$ an open neighbourhood $U \subset M$ containing x and a diffeomorphism $\phi_U : U \times R^m \rightarrow \pi^{-1}(U)$ such that the following diagram commutes

$$\begin{array}{ccc} U\times\mathbb{R}^m & \xrightarrow{\phi_U} & \pi^{-1}(U) \\ & \searrow \quad \swarrow^{\pi} & \\ & U & \end{array}$$

where the lefthand arrow is the projection on the first factor, and such that ϕ_U induces for every $y \in U$ an isomorphism $\{y\} \times \mathbb{R}^m \to \pi^{-1}(y)$ of real vectorspaces.

3.2 Constructing vectorbundles.
The definition given above is not always particularly easy to assimilate. It simply means that a vectorbundle over M is obtained by taking an open covering $\{U_i\}$ of M and gluing together products $U_i \times \mathbb{R}^m$ by means of diffeomorphisms which are linear (i.e. vectorspace structure preserving) in the second coordinate. Thus an m-dimensional vectorbundle over M is given by the following data

- an open covering $\{U_\alpha\}_{\alpha\in I}$ of M.
- for every α,β a smooth map $\phi_{\alpha\beta}: U_\alpha \cap U_\beta \to GL_m(\mathbb{R})$ where $GL_m(\mathbb{R})$ is the space of all invertible real $m\times m$ matrices considered as an open subset of $\mathbb{R}^m$. These data are subject to the following compatibility conditions
- $\phi_{\alpha\alpha}(x) = I_m$, the identity matrix, for all $x \in U_\alpha$
- $\phi_{\beta\gamma}(x)\phi_{\alpha\beta}(x) = \phi_{\alpha\gamma}(x)$ for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$

From these data E is constructed by taking the disjoint union of the $U_\alpha \times \mathbb{R}^m$, $\alpha\in I$ and identifying $(x,v) \in U_\alpha \times \mathbb{R}^m$ with $(y,w) \in U_\beta \times \mathbb{R}^m$ if and only if $x=y$ and $\phi_{\alpha\beta}(x)v = w$. The morphism π is induced by the first coordinate projections $U_\alpha \times \mathbb{R}^m \to U_\alpha$.

3.3 Constructing vectorbundles 2.
If the base manifold M is itself viewed as a smoothly glued together collection of open sets in $\mathbb{R}^n$ we can describe the gluing for M and for the vectorbundle all at once. The combined data are then as follows

- open sets $U_\alpha \times \mathbb{R}^m$, $U_\alpha \subset \mathbb{R}^n$ for all $\alpha\in I$
- open subsets $U_{\alpha\beta} \subset U_\alpha$ for all $\alpha,\beta \in I$
- diffeomorphsims $\phi_{\alpha\beta}: U_{\alpha\beta} \to U_{\beta\alpha}$
- diffeomorphisms $\tilde{\phi}_{\alpha\beta}: U_{\alpha\beta} \times \mathbb{R}^m \to U_{\beta\alpha} \times \mathbb{R}^m$ of the form $(x,v) \mapsto (\phi_{\alpha\beta}(x), A_{\alpha\beta}(x)v)$ where $A_{\alpha\beta}(x)$ is an $m\times m$ invertible real matrix depending smoothly on x.

These data are then subject to the same compatibility conditions for the $\tilde{\phi}_{\alpha\beta}$'s (and hence the $\phi_{\alpha\beta}$) as described in 2.4 above.

Again, as in the case of differentiable manifolds, it is sometimes a good idea to view a vectorbundle $\pi: E \to M$ simply as a collection of local pieces $\pi_\alpha: U_\alpha \times \mathbb{R}^m \to U_\alpha$ together with gluing data (transformation rules).

3.4 Example: the tangent vectorbundle of a smooth manifold.
Let the smooth manifold M be given by the data U_α, $U_{\alpha\beta}$, $\phi_{\alpha\beta}$ as in 2.4. Then the tangent bundle TM is given by the data

- $U_\alpha \times \mathbb{R}^n$, $U_{\alpha\beta} \times \mathbb{R}^n \subset U_\alpha \times \mathbb{R}^n$
- $\tilde{\phi}_{\alpha\beta}: U_{\alpha\beta} \times \mathbb{R}^n \to U_{\beta\alpha} \times \mathbb{R}^n$, $\tilde{\phi}_{\alpha\beta}(x,v) = (\phi_{\alpha\beta}(x), J(\phi_{\alpha\beta})(x)v)$

where $J(\phi_{\alpha\beta})(x)$ is the Jacobian matrix of $\phi_{\alpha\beta}$ at $x \in U_{\alpha\beta}$.

Exercise: check that these gluing morphisms do indeed define a vectorbundle; i.e. the compatibility. (This is the chain rule!)

3.5 Example. The canonical bundle over a Grassmann manifold.
As said above, intuitively a vectorbundle over M is a family of vectorspaces smoothly parametrized by M. I.e. for each $x \in M$ there is given a vectorspace V_x, the fibre over x and the V_x vary smoothly with x. In this intuitive fashion the canonical bundle over $Gr_k(\mathbb{R}^n)$, the space of k dimensional subspaces of $\mathbb{R}^n$, is the bundle whose fibre over $x \in Gr_k(\mathbb{R}^n)$ "is" the vectorspace x.

In terms of gluing data, and more precisely, this vectorbundle is described as follows. Recall that $Gr_k(\mathbb{R}^n)$ was obtained from local pieces $U_\alpha \simeq \mathbb{R}^{k(n-k)}$

$$U_\alpha = \{A \in \mathbb{R}^{k \times n} : A_\alpha = I_k\}$$

Now define

$$\tilde{\phi}_{\alpha\beta}: U_{\alpha\beta} \times \mathbb{R}^k \to U_{\beta\alpha} \times \mathbb{R}^k$$
$$(A,v) \to (A_\beta^{-1}A, (A_\beta)^T v)$$

It is again the same observation that $(SA)_\alpha = SA_\alpha$ which proves the compatibility relation $\tilde{\phi}_{\beta\gamma} \circ \tilde{\phi}_{\alpha\beta} = \tilde{\phi}_{\alpha\gamma}$.

This bundle is the universal k-dimensional vectorbundle over $Gr_k(\mathbb{R}^n)$ as usually defined by topologists. The algebraic geometers often prefer to work with the dual object: the bundle over $Gr_k(\mathbb{R}^n)$ whose fibre over x is the quotient space $\mathbb{R}^n/x$. Exercise: give a gluing data description of this last bundle.

3.5 Morphisms of vectorbundles.
A morphism of vectorbundles from the vectorbundle $\pi: E \to M$ to the vectorbundle $\pi': E' \to M'$ is a pair of smooth maps $\tilde{f}: E \to E', f: M \to M'$ such that $\pi' \circ \tilde{f} = f \circ \pi$ and such that the induced map $\tilde{f}_x: \pi^{-1}(x) \to \pi^{-1}(f(x))$ is homomorphism of vectorspaces for all $x \in M$. We leave it to the reader to translate this into a local pieces and gluing data description.

As an example consider two manifolds M,N both described in terms of local pieces and gluing data. Let $f: M \to N$ be given in these terms by the $f_{\alpha\beta}: U_{\alpha\beta} \to V_\beta$ (cf 2.10 above). Then the maps $\tilde{f}_{\alpha\beta}: U_{\alpha\beta} \times \mathbb{R}^n \to V_\beta \times \mathbb{R}^m$ defined by $\tilde{f}_{\alpha\beta}(x,v) = (f_{\alpha\beta}(x), J(f_{\alpha\beta})(x)v)$ combine to define a morphism of vectorbundles $\tilde{f} = Tf: TM \to TN$.

4. VECTORFIELDS

A vectorfield on a manifold M assigns in a differentiable manner to every $x \in M$ a tangent vector at x, i.e. an element of the fibre $T_xM = \pi^{-1}(x)$ of the tangent bundle TM. Slightly more precisely this gives the

4.1 Definitions.
Let $\pi: E \to M$ be a vectorbundle. Then a section of E is a smooth map $s: M \to E$ such that $\pi \circ s = id$. A section of the tangent vectorbundle $TM \to M$ is called a vectorfield.

Suppose that M is given by a local pieces and gluing data description as in 2.4 above. Then a vectorfield s is given by "local sections" $s'_\alpha: U_\alpha \to U_\alpha \times \mathbb{R}^n$ of the form $s'_\alpha(x) = (x, s_\alpha(x))$, i.e. by a collection of functions $s_\alpha: U_\alpha \to \mathbb{R}^n$ such that $J(\phi_{\alpha\beta})(x)(s_\alpha(x)) = s_\beta(\phi_{\alpha\beta}(x)$ for all $x \in U_{\alpha\beta}$.

4.2 Derivations.
Let A be an algebra over $\mathbb{R}$. Then a derivation is an $\mathbb{R}$-linear map $D: A \to A$ such that $D(fg) = (Df)g + f(Dg)$ for all $f, g \in A$.

4.3 Derivations and vectorfields.

Now let M be a differentiable manifold and let $S(M)$ be the $\mathbb{R}$-algebra of smooth functions $M \to \mathbb{R}$. Then every vectorfield s on M defines a derivation of $S(M)$, (which assigns to a function f its derivative along s), which can be described as follows. Let M be given in terms of local pieces U_α and gluing data $U_{\alpha\beta}$, $\phi_{\alpha\beta}$. Let $f:M \to \mathbb{R}$ and the section $s:M \to TM$ given by the local functions $f_\alpha : U_\alpha \to \mathbb{R}$, $s_\alpha : U_\alpha \to \mathbb{R}^n$. Now define $g_\alpha : U_\alpha \to \mathbb{R}$ by the formula

$$g_\alpha(x) = \sum_k s_\alpha(x)_k \frac{\partial f_\alpha}{\partial x_k}(x) \tag{4.4}$$

where $s_\alpha(x)_k$ is the k-th component of the n-vector $s_\alpha(x)$. It is now an easy exercise to check that $g_\beta(\phi_{\alpha\beta}(x)) = g_\alpha(x)$ for all $x \in U_{\alpha\beta}$ (because $J(\phi_{\alpha\beta})(x)s_\alpha(x) = s_\beta(\phi_{\alpha\beta}(x))$ for these x) so that the $g_i(x)$ combine to define a function $g = D_s(f):M \to \mathbb{R}$. This defines a map $D:S(M) \to S(M)$ which is seen to be a derivation. Inversely every derivation of $S(M)$ arises in this way.

4.5 The Lie bracket of derivations and vectorfields.

Let D_1, D_2 be derivations of an $\mathbb{R}$-algebra A. Then, as is easily checked, so is

$$[D_1, D_2] = D_1 D_2 - D_2 D_1$$

So if s_1, s_2 are vectorfields on M, then there is a vectorfield $[s_1, s_2]$ on M corresponding to the derivation $[D_{s_1}, D_{s_2}]$. This vectorfield is called the Lie bracket of s_1 and $(s_1, s_2) \mapsto [s_1, s_2]$ defines a Lie algebra structure on the vectorspace $V(M)$ of all vectorfields on M.

If M is given in terms of local pieces U_α and gluing data $U_{\alpha\beta}$, $\phi_{\alpha\beta}$ then the Lie bracket operation can be described as follows. Let the vectorfields s and t be given by the local functions $s_\alpha, t_\alpha : U_\alpha \to \mathbb{R}^n$ with components s_α^i, t_α^i, $i = 1, \ldots, n$. Then $[s,t]$ is given by the local functions

$$[s,t]_\alpha^i = \sum_j s_\alpha^j \frac{\partial t_\alpha^i}{\partial x_j} - \sum_j t_\alpha^j \frac{\partial s_\alpha^i}{\partial x_j}$$

4.6 The $\frac{\partial}{\partial x}$ notation.

Let the vectorfield $s:M \to TM$ be given by the functions $s_\alpha : U_\alpha \to \mathbb{R}^n$. Then, using the symbols $\frac{\partial}{\partial x_k}$ in first instance simply as labels for the coordinates in $\mathbb{R}^n$, we can write

$$s_i = \sum s_i(x)^k \frac{\partial}{\partial x_k} \tag{4.7}$$

This is a most convenient notation because as can be seen from (4.4) this gives precisely the local description of the differential operator (derivation) D_s associated to s.

Further taking the commutator difference of the two (local) differential operators

$$D_s = \sum_k s^k \frac{\partial}{\partial x_k}, \quad D_t = \sum_l t^l \frac{\partial}{\partial x_l}$$

gives

$$(\sum_k s^k \frac{\partial}{\partial x_k})(\sum_l t^l \frac{\partial}{\partial x_l})(f) = \sum_{k,l} s^k t^l \frac{\partial^2 f}{\partial x_k \partial x_l} + \sum_{k,l} s^k \frac{\partial t^l}{\partial x_k} \frac{\partial f}{\partial x_l}$$

$$(\sum_l t^l \frac{\partial}{\partial x_l})(\sum_k s^k \frac{\partial}{\partial x_k})(f) = \sum_{k,l} t^l s^k \frac{\partial^2 f}{\partial x_l \partial x_k} + \sum_{k,l} t^l \frac{\partial s^k}{\partial x_l} \frac{\partial f}{\partial x_k}$$

so that

$$[D_s, D_t]f = (D_s D_t - D_t D_s)f = (\sum_{k,l} (s^k \frac{\partial t^l}{\partial x_k} - t^k \frac{\partial s^l}{\partial x_k}) \frac{\partial}{\partial x_l})(f)$$

which fits perfectly with the last formula of 4.5 above.

Finally a substitution $y=\phi(x)$ in a differential operator (4.7) transforms it precisely according to the same rule as applies to the corresponding vectorfield s, cf the last formula of 4.1 above.

4.8 Differential equations on a manifold.
A differential equation on a manifold M is given by an equation

$$\dot{x} = s(x) \tag{4.9}$$

where $s:M\to TM$ is a vectorfield, i.e. a section of the tangentbundle. At every moment t, equation (4.8) tells us in which direction and how fast $x(t)$ will evolve by specifying a tangent vector $s(x(t))$ at $x(t)$.

Again it is often useful to take a local pieces and gluing data point of view. Then the differential equation (4.8) is given by a collection of differential equations $\dot{x}=s_i(x)$ in the usual sense of the word on U_i where the functions $s_\alpha(x)$ satisfy $J(\phi_{\alpha\beta}(x)s_\alpha(x)=s_\beta(\phi_{\alpha\beta}(x))$ for all $x\in U_{\alpha\beta}$.

In these terms a solution of the differential equation is simply a collection of solutions of the local equations, i.e. a collection of maps $f_\alpha:V_\alpha\to U_\alpha, V_\alpha\subset\mathbb{R}(\geq 0)$ such that $\cup V_\alpha=\mathbb{R}(\geq 0)$, $\frac{d}{dt}f_\alpha(t)=s_\alpha(f_\alpha(t))$ which fit together to define a morphism $\mathbb{R}(\geq 0)\to M$, i.e. such that $\phi_{\alpha\beta}(f_\alpha(t))=f_\beta(t)$ if $t\in V_\alpha\cap V_\beta$.

In more global terms a solution of (4.8) which passes through x_0 at time 0 is a morphism of smooth manifolds $f:\mathbb{R}\to M$ such that $Tf:T\mathbb{R}\to TM$ satisfies $Tf(t,1)=s(f(t))$ for all $t\in\mathbb{R}$ (or a suitable subset of $\mathbb{R}$), i.e. Tf takes the vectorfield $1:\mathbb{R}\to T\mathbb{R} = \mathbb{R}\times\mathbb{R}$, $t\mapsto(t,1)$ into the vectorfield (section) $s:M\to TM$.

4.10 Example. The matrix Riccati differential equation.
The simplest Riccati equation is

$$\dot{x} = 1-x^2 \tag{4.11}$$

This one has finite escape time. Indeed an initial value of $x(0)<-1$ gives a finite escape time. For this one it is still easy to figure out what happens near infinity and whether and how the trajectory goes through infinity and comes back. The general matrix Riccati equation is

$$\dot{K} = KA + KBK + C + DK \tag{4.12}$$

where K is an $m\times n$ matrix and A,B,C,D are known constant matrices of sizes $n\times n, n\times m, m\times n$ and $m\times m$ respectively. This one is very hard to understand directly. The first step of a somewhat indirect approach is as follows. Consider $n\times(n+m)$ matrices partitioned into two blocks of sizes $m\times n$ $m\times m$ respectively. Now consider the linear system of equations

$$\frac{d}{dt}(X\ \ Y) = (XY)P,\ \ P=\begin{pmatrix} A & -B \\ C & -D\end{pmatrix} \tag{4.13}$$

Let $K=Y^{-1}X$ (assuming for the moment that Y^{-1} exists). Then

$$\frac{d}{dt}K = -Y^{-1}\dot{Y}Y^{-1}X+Y^{-1}\dot{X}=Y^{-1}(XB+YD)Y^{-1}X+Y^{-1}(XA+YC)$$

$$= KBK + DK + KA + C$$

In other words the matrix Riccati equation lifts to a linear equation on $\mathbb{R}^{n\times(n+m)}$. If $(X(0)\ Y(0))$ is a full rank matrix, then so is $(X(t)\ Y(t))=(X(0)\ Y(0)e^{tP}$ for all t. But even so $Y(t)$ may very well become noninvertible and that accounts for finite escape time phenomena of the Riccati equation. As already noted if $K(0)\in\mathbb{R}^{m\times(n+m)}_{reg}$ then K remains in this subspace. Now we have already seen the projection map

$$\pi:\mathbb{R}_{reg}^{m\times(n+m)} \to Gr_m(\mathbb{R}^{n+m})$$

in section 2.12 above. I claim that the differential equation (4.13) descends (i.e. induces) a differential equation on the manifold $Gr_m(\mathbb{R}^{n+m})$. To prove this one must show that if $M=(X,Y)$ and $M_1=(X_1,Y_1)$ in $\mathbb{R}_{reg}^{mx\times(n+m)}$ both map to the same point $x\in Gr_m(\mathbb{R}^{n+m})$ then $T\pi$ takes the respective tangent vectors at M and M_1 into the same tangent vector at x. This is essentially the same calculation as we already did (several times). Indeed let $x\in U_\alpha$. The map π is given locally by $M\mapsto M_\alpha^{-1}M$. Now $M(t)=Me^{tP}$ and $M_1(t)=M_1e^{tP}$. Now if M and M_1 both map to the same $x\in Gr_m(\mathbb{R}^{n+m})$ then $M_1=SM$ for some constant matrix S. But then

$$M_1(t) = M_1e^{tP}=SMe^{tP}=SM(t)$$

for all t. So that $M_1(t)$ and $M(t)$ map to the same point $x(t)\in Gr_n(\mathbb{R}^{n+m})$ for all t. This proves the claim. However $Gr_m(\mathbb{R}^{n+m})$ is a smooth *compact* manifold (a fact I did not prove), so $x(t)\in Gr_m(\mathbb{R}^{n+m})$ of all t. The finite escape time phenomena of the matrix Riccati are now analyzed and understood in terms of the embedding

$$K \mapsto m-\text{dim subspace of } \mathbb{R}^{n+m} \text{ spanned by the rows of } (K \ \ I_m)$$

$$\mathbb{R}^{m\times n} \hookrightarrow Gr_m(\mathbb{R}^{n+m})$$

The matrix Riccati equation is at first only defined on $\mathbb{R}^{m\times n}$. It extends to a equation on the smooth compactification $Gr_m(\mathbb{R}^{n+m})$ (but not to other compactifications such as the projective space $\mathbb{P}^{mn}$ (unless $m=1$) or the sphere S^{nm}). From time to time $x(t)$ may exit from the open dense subset $\mathbb{R}^{m\times n}$ in $Gr_m(\mathbb{R}^{n+m})$ to cross the set at infinity $Gr_m(\mathbb{R}^{n+m})\setminus\mathbb{R}^{m\times n}$.

4.14 Compatibility of vectorfields under differentiable maps.
Let $\phi:M\to N$ be a map of differentiable manifolds. Then, as we have seen, 3.5, for each $x\in M$ we have the induced map $T\phi(x):T_xM\to T_{\phi(x)}N$ of the tangent vectorspace of M at x to the tangent vectorspace of N at $\phi(x)$. All together these map define the vectorbundle map $T\phi:TM\to TN$ which is also often denoted $d\phi$.

Now let $\alpha:M\to TM$ be a vectorfield on M, $\alpha(x)\in T_xM$. Then we have the various tangent vectors $T\phi(x)(\alpha(x))\in T_{\phi(x)}N$. These may or may not define a vectorfield on N. Firstly because not every $y\in N$ need to be of the form $\phi(x)$ for some $x\in M$ and secondly because if x and x' both map to the same $y\in N$ then it may very well happen that $T\phi(x)(\alpha(x))\neq T\phi(x')(\alpha(x'))$. (As well shall see below the dual notion to that of a vectorfield, i.e. the notion of a differentiable 1-form, is much better behaved in this respect: for each 1-form ω on M and differentiable map $\phi:M\to N$ there is a canonically associated (induced) 1-form $\phi^*\omega$ on M.)

If two vectorfields α on M and β on N are such that $T\phi(x)(\alpha(x))$ for all $x\in M$ then we say that α and β are compatible under ϕ. If $\phi:M\to N$ is a diffeomorphism then $\beta(y) = T\phi(\phi^{-1}(y))\alpha(\phi^{-1}(y))$ defines a unique vectorfield on N compatible with α on M under ϕ.

The Lie bracket of vectorfields $[\alpha,\alpha']$ is 'functorial' with respect to transforms of vectorfields in the following sense.

4.15 Proposition. Let α,β and α',β' be compatible pairs of vectorfields under $\phi:M\to N$. Then $[\alpha,\alpha']$ *and* $[\beta,\beta']$ *are also compatible under ϕ.*

The easiest way to see this is first to do the following exercise. Let $\phi:M\to N$ be differentiable and let the vectorfield α on M and β on N be compatible under ϕ. Let D_α be the derivation on $\mathfrak{F}(M)$ corresponding to α and D_β the one on $\mathfrak{F}(N)$ corresponding to β. (Here $\mathfrak{F}(M)$ is the ring of smooth functions on M.) Then for all $f\in\mathfrak{F}(N)$

$$D_\alpha(f\circ\phi) = D_\beta(f)\circ\phi \qquad (*)$$

Indeed in local coordinates x on M and y on N and ϕ given by $y_j=\phi_j(x)$

$$D_\alpha(f\circ\phi)(x) = \sum_i \alpha_i\frac{\partial}{\partial x_i}(f\circ\phi)(x) =$$

$$\sum_{i,j}\alpha_i\frac{\partial f}{\partial y_j}(\phi(x))(T\phi(x))_{ji} = \sum\beta_j\frac{\partial f}{\partial y_j}(\phi(x)) = (D_\beta f)(\phi(x)) = (D_\beta f\circ\phi)(x).$$

And inversely if (*) holds then α and β are compatible under ϕ. (This is the chain rule of course.) Now let α,β and α',β' be compatible pairs. Then by the exercise we just did

$$D_\alpha{}'D_\alpha(f\circ\phi) = D_\alpha{}'(D_\beta(f)\circ\phi) = D_\beta{}'D_\beta(f)\circ\phi$$

Thus $[D_\alpha{}',D_\alpha](f\circ\phi) = [D_\beta{}',D_\beta](f)\circ\phi$ so that the vectorfields belonging to $[D_\alpha{}',D_\alpha]$, i.e. $[\alpha',\alpha]$, and $[D_\beta{}',D_\beta]$, i.e. $[\beta',\beta]$, are also compatible.

4.16 Distributions.

A distribution Δ on a manifold specifies for each x a subspace $\Delta_x\subset T_xM$ of the tangent bundle at x. They arise naturally in several contexts. E.g. in control systems of the following kind

$$\dot{x} = \sum_{i=1}^{m}u_ig_i(x) \tag{4.17}$$

where the $u_i\in\mathbf{R}$ are controls (and the $g_i(x)$ are vectorfields). The corresponding distribution is (of course) defined by $\Delta_x=$ linear subspace of T_xM spanned by the tangent vectors $g_1(x),\ldots,g_m(x)$. In this setting Δ_x responds the totality of directions in which the state x can be made to move infinitesimally by suitable (constant in time) control vectors $(u_1,\ldots,u_m)$.

This does not mean that by taking suitable functions $u_1(t),\ldots,u_m(t)$ the vector x can not be made to move in still more directions, as we shall see immediately below. To get some feeling for this consider the special case

$$\dot{x} = \sum_{i=1}^{m}u_iA_ix\,,\quad x\in\mathbf{R}^n \tag{4.18}$$

where the A_i are constant $n\times n$ matrices. As is well known the solution of $\dot{x}=Ax$ is $x(t)=e^{At}x(0)$, where

$$e^{At} = I+\frac{At}{1!}+\frac{A^2t^2}{2!}+\cdots$$

Now let us take in (4.18)

$$u_1 = 1,\ u_2=\cdots=u_m=0,\ \text{for } t\in[0,\epsilon)$$
$$u_2 = 1,\ u_1=u_3=\cdots=u_m=0,\text{for } t\in[\epsilon,2\epsilon)$$
$$u_1 = -1,\ u_2=\cdots u_m=0,\ \text{for } t\in[2\epsilon,3\epsilon)$$
$$u_2 = -1,\ u_1=u_3=\cdots=u_m=0,\ \text{for } t\in[3\epsilon,4\epsilon]$$

then at time $t=4\epsilon$, we have

$$x(t) = e^{-A_2\epsilon}e^{-A_1\epsilon}e^{A_2\epsilon}e^{A_1\epsilon}x(0)$$

which is equal to

$$x(t) = x(0)+\epsilon^2(A_2A_1-A_1A_2)x(0)+O(\epsilon^3)$$

Thus from $x(0)$, $x(t)$ can also be made to move in the direction

$$[A_2,A_1]x(0) = (A_2A_1-A_1A_2)x(0)$$

Now, as follows from the formula at the end of (4.5) the vectorfield

$$(A_2A_1-A_1A_2)x$$

is precisely the Lie bracket of the two vectorfields A_1x, A_2x

$$[A_1x,A_2x] = (A_2A_1 - A_1A_2)x.$$

(Note the reversal of order with respect to the usual commutator difference of matrices).

The same holds for arbitrary vectorfields, i.e. for equations like (4.17): in addition to the immediately given directions $g_i(x), \ldots, g_m(x)$ the control system can be made to evolve in the directions $[g_i,g_j](i,j=1,\ldots,m)$, and $[g_i,[g_j,g_k]](i,j,k=1,\ldots,m)$, and $[[g_i,g_j],[g_k,g_l]](i,j,k,l=1,\ldots,m)$, etc., etc.; that is in all directions which can be constructed by taking repeated Lie brackets of the given vectorfields $g_1, \ldots, g_m$.

This leads to the notion of an involutive distribution. A vectorfield $s:N \rightarrow TM$ is said to belong to (or be in) the distribution Δ if $s(x) \in \Delta_x$ for all x. A distribution Δ is said to be *involutive* if for all vectorfields s and t in Δ the vectorfield $[s,t]$ is also in Δ.

Natural examples of involutive distributions arise as follows. A *foliation* of codimension r of M is a decomposition of M into subsets (called *leaves*) such that locally the decomposition looks like the decomposition $\mathbb{R}^n = \bigcup_{a \in \mathbb{R}^r} a + \mathbb{R}^{n-r}$ where $\mathbb{R}^r$ is viewed as the subvectorspace of $\mathbb{R}^n$ of vectors whose last $n-r$ coordinates are zero, and $\mathbb{R}^{n-r} \subset \mathbb{R}^r$ is the subspace of vectors whose first r coordinates are zero. Here the phrase 'locally looks like' means that for each $x \in M$, there is an open neigbourhood U of x and a diffeomorphism of U to an open subset V of $\mathbb{R}^n$ such that ϕ applied to the decomposition of U gives a decomposition of V as given above.

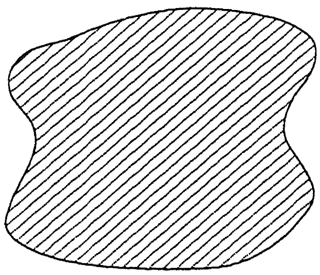

FIGURE 10 Foliation

Thus for codimension 1 a foliation locally looks like the picture of fig. 10. Note that each leaf is a submanifold. However the topology induced by M on the leaf need not be that of the leaf as a manifold in its own right. It is very possible that a leaf returns to a neigbourhood infinitely often. A foliation of M naturally defines an involutive distribution. Indeed for $x \in M$ let $F(x)$ be the leaf through x. If s and t are two vectorfields on $F(x)$ then so is $[s,t]$. Hence if Δ_x is the subspace of T_xM of all vectors tangent to $F(x)$ then Δ is an involutive distribution.

The converse is also true. If Δ is involutive and Δ_x is of constant rank, i.e. $\dim\Delta_x = k$ for all x, then there exists a foliation $\mathcal{F}$ of M such that Δ_x is the tangent space to $F(x)$ for all x. This is *Frobenius' theorem* and it is a sort of multi-time or multi-control variant of the existence of solutions theorem for ordinary differential equations.

5. RIEMANNIAN MANIFOLDS

A differentiable manifold as defined above is still a rather floppy (topological) structure. To have real fun and do real analysis, including stochastic analysis, some more structure is needed. One of the more popular is a Riemannian structure. Intuitively this means that each tangent space T_xM is provided with an inner product and these inner products are supposed to vary smoothly with x. As usual this is made precise by providing a local pieces and patching data description. Locally the manifold and its tangent bundle look like $U_\alpha \times \mathbb{R}^n, U_\alpha \subset \mathbb{R}^n$. Let P be the space of positive definite inner products, i.e. positive definite symmetric matrices. Then a Riemannian structure on M is given in terms of local data by a collection of smooth maps $g(\alpha): U_\alpha \to P$ with the following transformation properties: if $x \in U_{\alpha\beta}$ then

$$g(\beta)(\phi_{\alpha\beta}(x)) = (J(\phi_{\alpha\beta}(x))^T)^{-1} g(\alpha)(x) J(\phi_{\alpha\beta})(x)^{-1}$$

Because P is convex it is not difficult to see that these always exist Riemannian matrices. On a Riemannian manifold one can define the length and energy of a curve and one can relate the tangent vectors at one point of M to those at another point thus making all kinds of analysis and estimates possible.

6. Calculus

So far we have mainly dealt with the topology of manifolds, i.e. those gadgets which locally look like $\mathbb{R}^n$ and we have done a bit of differential calculus. E.g. if $f: M \to N$ is a differentiable mapping of M into N we know what the 'derivative' of f is, viz. the mapping $Tf: TM \to TN$ of the tangent bundle TM of M into the tangent bundle TN. And indeed if $x \in M$, then $Tf(x): T_x M \to T_x N$ is the linear part (approximation) of $f: x \mapsto f(x)$ at x.

Naturally we would also like to do the integral bit, that is to give the right kind of meaning to such things as the integral of a function on the sphere, say, over that sphere. This requires some more preparations having mainly to do with 'what (variables) to integrate against' or, more generally, 'what can be integrated over what'. Also, as we shall see, to integrate functions one needs more structure than just a manifold; e.g. a Riemannian metric will do.

6.1. Chains and cubes.

What we want to do is to define integrals over (broken) curves, surfaces etc. in arbitrary manifolds. Curves and surfaces etc. can be thought of as made up from pieces which are images of intervals, filled squares, filled cubes, etc. It turns out to be convenient to define integrals initially as 'something over a map of an interval, square, ... into M' rather than as something over the image of that map.

The standard n-cube Δ_n is $[0,1]^n \subset \mathbb{R}^n$ e.g. the square or the familiar 3-cube depicted below. The boundary of Δ_n is made up of various pieces isomorphic (but not identical) with Δ_{n-1}. More precisely for each $i = 1,\dots,n$ we define two maps $\alpha_0^i: \Delta_{n-1} \to \Delta_n$, $\alpha_1^i: \Delta_{n-1} \to \Delta_n$ as follows

$$\alpha_0^i(x_1,\dots,x_{n-1}) = (x_1,\dots,x_{i-1},0,x_i,\dots,x_{n-1})$$

$$\alpha_1^i(x_1,\dots,x_{n-1}) = (x_1,\dots,x_{i-1},1,x_i,\dots,x_{n-1})$$

The images of these maps make up the boundary of Δ_n.

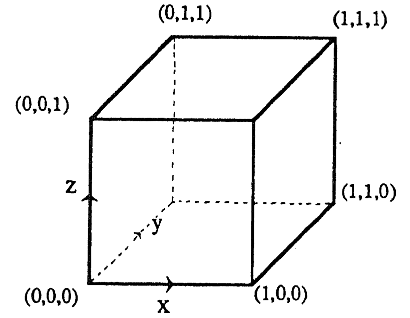

FIGURE 11

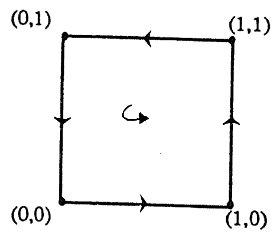

FIGURE 12

A *singular n-cube* in a subspace M of some $\mathbb{R}^m$ is a mapping $s: \Delta_n \to M$. (Here it is good to think of m as larger of equal to n.) A *singular n-chain* is a finite formal sum $\sum n_i s_i$ where the n_i are integers and the s_i are singular n-cubes. The boundary ∂s of a singular n-cube $s: \Delta_n \to A$ is by definition the $(n-1)$-chain $\partial s = \sum_{i=1}^{n} (-1)^i (s \circ \alpha_0^i - s \circ \alpha_1^i)$. Thus the boundary of the 3-cube $\mathrm{id}: \Delta_3 \to \Delta_3 \subset \mathbb{R}^n$ is (in terms of images of Δ_2) equal to + (front square) - (back square) - (left square) + (right square) - (bottom) + (top), and the boundary of $\text{id}:\Delta_2 \to \Delta_2 \subset \mathbb{R}^2$ is the sum of intervals: $[(0,0),(1,0)]+[(1,0),(1,1)]-[(0,1),(1,1)]-[(0,0),(0,1)]$ which fits our intuitive idea of the (oriented) boundary of the square. The boundary of a singular n-chain $c \cong \sum n_i s_i$ is by definition equal to $\partial c = \sum n_i \partial s_i$.

These are the formal definitions. In practices one tends to think of a singular n-chain in terms of the images (with multiplicities) of the singular n-cubes making up the chain as illustrated below. Intuitively the boundary of the piece of surface (corresponding to the chain s_1+s_2) depicted in fig. 13 ought to be the outer circle. And if s_1 and s_2 are chosen such that maps $\Delta_1 \to M$ induced by s_1 and s_2 for the piece of boundary in the middle are the same then this will indeed be the case (thanks to the orientations chosen). Moreover s_1 and s_2 can always be chosen in such a way. However if s_1 and s_2 are just any differentiable maps whose images happen to fit together as indicated, then the boundary of s_1+s_2 will be more complicated. It turns out that for integration purposes (and the multidimensional generalization of the fundamental theorem of calculus: Stokes theorem) this matters little.

For clarities sake let us remark that $\Delta_0 = [0,1]^0$ is a single point and that, thus, a singular 0-chain in M is just a finite set of points in M with multiplicities.

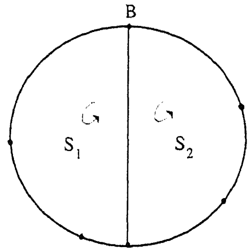

FIGURE 13

6.3. Forms

The next thing to decide is what kinds of animals can be integrated. As everyone knows functions cannot be quite the right answer. Simply because under a change of variables the things under an integral sign do not transform as functions. Indeed if $\phi:\mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism (change of variables $y=\phi(x)$). Then $f:\mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x)$ transforms as $y \mapsto f(\phi^{-1}(y))$, i.e. $f \mapsto f \circ \phi^{-1}$. But for an integral we have

$$\int_A f = \int_{\phi(A)} f(\phi^{-1}(y))\,|\det J(\phi)(\phi^{-1}(y))|^{-1}$$

which is, of course, the reason one writes fdx or something like that under an integral sign.

The kinds of things which belong under an integral sign turn out to be *differential forms*. These things we now proceed to define.

If V is a vectorspace, a k-form on V is a k-multilinear mapping

$$\omega: V\times \cdots \times V \to \mathbb{R}$$

such that moreover for each $i \neq j$,

$$\omega(\cdots, v_i, \cdots, v_j, \cdots) = -\omega(\cdots, v_j, \cdots, v_i, \cdots)$$

i.e. interchanging two arguments just causes a sign change. One particular n-form on $\mathbb{R}^n$ is very well-known, the determinant, where $\det(v_1,\ldots,v_n)$, $v_i \in \mathbb{R}^n$ is the determinant of the $n\times n$ matrix obtained by writing out the n-vectors $v_1,...,v_n$ as column vectors in the standard basis. It is moreover just about unique as most everyone knows: if ω is a n-form on $\mathbb{R}^n$ then $\omega =$ a det for some constant $a \in \mathbb{R}$.

More generally the space $\Omega^k(V)$ of k-forms on a n-dimensional vectorspace V has dimension $\binom{n}{k}$. It will be useful to have a basis for $\Omega^k(V)$. Let $e_1,...,e_n$ be a basis for V, and let $\phi_1,...,\phi_n$ be the dual basis, i.e. $\phi_i(e_j)=\delta_{ij}$. Then a basis for $\Omega^k(V)$ is given by the functions $\phi_{i_1}\wedge \cdots \wedge\phi_{i_k}$, $i_1 < \cdots < i_k$ defined by the formula

$$(\phi_{i_1}\wedge\cdots\wedge\phi_{i_k})(e_{j_1},\cdots,e_{j_k}) = \begin{cases} 0 \text{ if } \{i_1,...,i_k\}\neq\{j_1,\cdots,j_k\} \\ \text{sign } \tau \text{ if there is a permutation } \tau \text{ of } i_1,...,i_k \\ \text{such that } \tau(i_r)=j_r, r=1,...,k \end{cases}$$

Thus for example $\phi_1\wedge\phi_2 \in \Omega^2(\mathbb{R}^3)$ takes the values

	(e_1,e_1)	(e_1,e_2)	(e_1,e_3)	(e_2,e_1)	(e_2,e_2)	(e_2,e_3)	(e_3,e_2)	(e_3,e_3)
$\phi_1\wedge\phi_2$	0	1	0	-1	0	0	0	0

It is useful to declare by definition that for arbitrary $i_1,...,i_k \in \{1,...,n\}$

$$\phi_{j_1}\wedge\cdots\wedge\phi_{j_k} = \text{sign}\,\sigma\,\phi_{i_1}\wedge\cdots\wedge\phi_{i_k}$$

if the $j_1,...,j_k$ are all different where $(i_1,...,i_k)$ is the unique permutation of $(j_1,...,j_k)$ such that $i_1<i_2<\cdots<i_k$ and $\sigma(j_k) = i_k$, and to set $\phi_{j_1}\wedge\cdots\wedge\phi_{j_k}=0$ if two or more of the ϕ_{j_r} are equal.

Now let M be a manifold. Then a differentiable k-form ω on M consists of giving an k-form $\omega(x)$ on T_xM for all x such that $\omega(x)$ varies smoothly with x. As usual this can be given a local and gluing data description. Let M be obtained by patching together pieces $U_\alpha \subset \mathbb{R}^n$ with the help of gluing functions $\phi_{\alpha\beta}$. On U_α the k-form ω is specified by giving functions

$$\omega_\alpha^{i_1\cdots i_k},\ i_1,...,i_k \in \{1,...,n\}$$

The corresponding form is then defined by

$$\omega_\alpha(v_1,...,v_k) = \sum_{i_1,...,i_k} \omega_\alpha^{i_1\cdots i_k} v_{1i_1}\cdots v_{ki_k}$$

where $v_j \in \mathbb{R}^n$ and v_{ji_j} is the i_j-th component of the vector v_j. For the $\omega_\alpha^{i_1\cdots i_k}$ to define a k-form i.e. an *alternating* k-multilinear function, it is necessary and sufficient that $\omega^{\cdots i\cdots j\cdots} = -\omega^{\cdots j\cdots i\cdots}$. Thus it suffices to specify the $\omega_\alpha^{i_1\cdots i_k}$ for $i_1<\cdots<i_k$. A collection of 'local' k-forms ω_α on U_α defines a k-form on all of M provided the ω_α are compatible in the sense that one must have

$$\omega_\beta(\phi(x))(J(\phi_{\alpha\beta}(x)v_1,...,J(\phi_{\alpha\beta}(x)v_k) = \omega_\alpha(x)(v_1,...,v_k)$$

for all $v_1,...,v_k \in T_xM$. This means that if $(s_{ij}) = J(\phi)(x)$

$$\omega_\alpha(x)^{i_1...i_k} = \sum_{j_1,...,j_k} \omega_\beta(\phi(x))^{j_1...j_k} s_{j_1i_1}...s_{j_ki_k} \tag{6.4}$$

Note the similarity with the compatibility requirements for the local pieces and gluing data description of a Riemannian metric. Indeed both are examples of contravariant tensors, g is a symmetric 2-tensor and ω is an alternating (= antisymmetric) k-tensor. For a local piece $U \subset \mathbb{R}^n$ of the manifold M let us use again the notation $\partial/\partial x_1,...,\partial/\partial x_n$ for the canonical basis of T_xU, $x \in U$. Let us use the symbols $dx_1,...,dx_n$ to denote the dual basis; i.e. $dx_i(\partial/\partial x_j)=\delta_{i,j}$. Then a differential k-form ω with components $\omega_\alpha^{i_1,...,i_k}$ can be written as

$$\omega_\alpha = \sum_{i_1<\cdots<i_k} \omega_\alpha^{i_1,...,i_k} dx_{i_1}\wedge\cdots\wedge dx_{i_k}$$

This is a rather good notation because (exercise!) it fits perfectly with the transformation rules (6.4).
Let $f:M \to N$ be a differentiable map of manifolds. Than there is an induced map $(Tf)(x):T_xM \to T_xN$ for all $x \in M$. Now let ω be a k-form on N i.e. for every $y \in N$ there is an antisymmetric k-linear mapping

$$\omega(y):T_yN \times \cdots \times T_yN \to \mathbb{R}$$

Then there is a natural k-form $f^*\omega$ on M defined by

$$f^*(\omega)(x)\colon T_xM \times \cdots \times T_xM \to \mathbb{R}$$
$$(v_1,\ldots,v_k) \mapsto \omega(f(x))((Tf)(x)v_1,\ldots,(Tf)(x)v_k)$$

6.4. Integrals 1.
The geometric preparations above are enough to enable us to make a first attempt at defining integrals. It turns out that what one can integrate is k-forms over k-chains. The first step is as follows.
Let ω be a k-form on the standard cube Δ_k. Then ω is given by a function f on Δ_k

$$\omega = f dx_1 \wedge \cdots \wedge dx_k$$

One now defines

$$\int_{\Delta_k} \omega = \int_{\Delta_k} f \tag{6.5}$$

where the right hand side of (6.5) is the usual Lebesgue integral. The next step is to define integrals over a singular cube. Thus let $s:\Delta_n \to M \subset \mathbb{R}^m$ be a smooth singular cube and let ω be a k-form on M. Then one defines

$$\int_s \omega = \int_{\Delta_k} s^*\omega \tag{6.6}$$

and for a singular k-chain $c = \sum n_i s_i$ one takes of course

$$\int_c \omega = \sum_i n_i \int_{s_i} \omega \tag{6.7}$$

Formula (6.6) defines an integral over each singular cube $s:\Delta_k \to M$. This is definitely not yet something like an integral over the subset $s(\Delta_k)$ of M. Nor can it be. For one thing if $k=1$ and the curve $s(\Delta_1)$ runs from A to B say, we definitely want the integral from A to B along the curve to be equal to minus the integral from B to A along the curve. This brings in the point of orientations, cf 6.8 below. For another if say $s':\Delta_1 \to M$ is defined by $s'(t) = s(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $s'(t) = B$ for $\frac{1}{2} \leq t \leq 1$, then, as is very easy to see, as a rule the integral over s' of ω will be different from the integral over s of ω. However as we shall see below for nice enough singular chains $c = \sum n_i s_i$, the integral of ω over c will only depend on the image of c understood in the sense of a family $s_i(\Delta_n)$ of twisted (smooth) cubes with multiplicities n_i and then one can truly speak of an integral of ω over the "subset" $c(\Delta_n)$ of M. Here nice enough will turn out to mean that each s_i must be orientation preserving and define a smooth imbedding $s_i:\Delta_n \to M$.

6.8. Orientations.
Consider all bases $(a_1,\ldots,a_n)$ of a vectorspace V of dimension n. We say that two bases $(a_1,\ldots,a_n)$ $(b_1,\ldots,b_n$ are in the same orientation class if the matrix (s_{ij}) defined by $b_j = \sum s_{ij} a_i$ has positive determinant. Thus there are two orientation classes often denoted + and -. Giving an orientation on V means specifying one of these classes which is often done by specifying one particular basis in that class. The usual ('counterclockwise in case $n=2$') orientation on $\mathbb{R}^n$ is given by the standard basis $(e_1,\ldots,e_n)$. An isomorphism $\phi:V \to W$ of oriented vectorspaces is orientation preserving if ϕ takes the

orientation class of V into that of W, i.e. if $(a_1,...,a_n)$ is a basis from the orientation class of V then $(f(a_1),...,f(a_n))$ must be a basis of the orientation class of W.

Let M be a manifold, thought of, as usual, as obtained by gluing together local pieces $U_\alpha \subset \mathbb{R}^n$. An orientation on M is now specified by choosing an orientation on each $x \times \mathbb{R}^n$, $x \in U_\alpha$, all α, such that

(i) if $x,y \in U_\alpha$, $x \times \mathbb{R}^n$ and $y \times \mathbb{R}^n$ have the same orientation (i.e. $(x,v) \mapsto (y,v)$ is orientation preserving.

(ii) if $x \in U_\alpha, y \in U_\beta$, $\phi_{\alpha\beta}(x)=y$ then $\det(J(\phi_{\alpha\beta}(x))>0$

This can not always be done. A classic example of a non-orientable manifold is the Möbius strip defined above. Exercise: prove this. Another example is the projective plane $\mathbb{P}^2_{\mathbb{R}} = Gr_1(\mathbb{R}^3)$.

A manifold together with an orientation is an *oriented manifold*. If $f:M \to N$ is a differentiable immersion of an oriented manifold M into an oriented manifold N of the same dimension then f is called *orientation preserving* if $Jf(x): T_xM \to T_{f(x)}N$ is orientation preserving for all $x \in M$. A smooth singular n-cube $s:\Delta_n \to M$, $\dim M = n$, is orientation preserving if there exists an extension of s to some open neighbourhood U of Δ_n in $\mathbb{R}^n$ such that this extension is orientation preserving.

6.9. Integrals 2.

Now consider an oriented submanifold N of dimension k of a manifold M. Let $c = \sum n_i s_i$, $c' = \sum n_i s_i'$ (same n_i) be two singular k-chains in N such that $s_i(\Delta_k) = s_i'(\Delta_k)$ for all i and such that both s_i and s_i' are orientation preserving for all i. Let ω be a k-form on M and (hence) on N. Then

$$\int_c \omega = \int_{c'} \omega$$

In particular if all the n_i are $+1$ and the images of the s_i fit together to define a piecewise differentiable submanifold with boundary N' of N as indicated in fig. 14 then we can truly speak of $\int_{N'} \omega$, the integral of ω over N'. To define this integral of course we first reduce to the case of one singular k-cube, cf above.

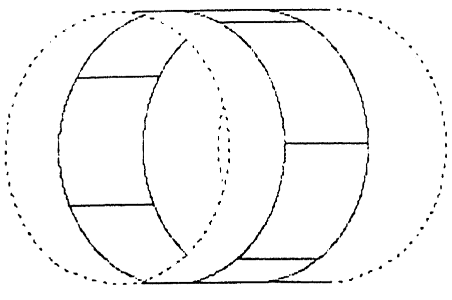

FIGURE 14

In that case we needed to assume $N \subset \mathbb{R}^m$ for some m, cf 6.4. Since every manifold can be embedded in an $\mathbb{R}^m$ for large enough m ($m \geq 2$ dim (manifold) $+1$ suffices) this is no real restriction. However, this does not fit well with our overriding attitude of viewing a manifold simply as a collection of local pieces U_α to be fitted together.

Let M be the manifold obtained by gluing the U_α; let $U_\alpha' \subset M$ be the piece corresponding to U_α. By cutting up Δ_k into smaller cubes if necessary we can see to it that the image of the chain c is such that it is made up of singular cubus which each lie completely into some coordinate neighbourhood U_α'. Then c is specified by a corresponding map $s':\Delta_k \to U_\alpha$ (such that the diagram of fig. 15 commutes) and the integral is defined entirely in terms of the local descriptions $s_\alpha:\Delta_k \to U_\alpha$.

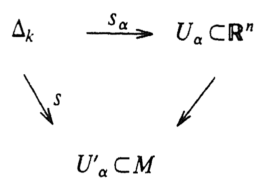

FIGURE 15

A zero-chain $c=\sum n_i P_i$ is a collection of points with multiplicities. A zero-form is a function $F:M \to \mathbb{R}$. The integral of a zero-form F over a zero-chain c is defined as $\sum n_i F(P_i)$.

6.10. The fundamental theorem of calculus.

The fundamental theorem of calculus (one variable) says that if F is a function with derivative $f=F'$ then

$$\int_a^b f dx = F(b)-F(a)$$

In our setting fdx is a one-form, F is a function, i.e. a zero-form. The 'chain' over which we integrate is an interval $[a,b]$ with boundary the 0-chain 'b'-'a' meaning the formal sum of 1 times the point b minus 1 times the point a. Thus the integral of $dF=fdx$ over $[a,b]$ is the integral of F over the boundary 'b'-'a'. This generalizes. To that end we need to define $d\omega$ of an arbitrary k-form ω. As usual let the manifold M be obtained by gluing together local pieces U_α and let the k-form ω be given locally by the ω_α

$$\omega_\alpha = \sum_{i_1 < \cdots < i_k} \omega_\alpha(x)^{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

In case $k=0$, ω is a function f and one defines the 1-form

$$df_\alpha = \sum_i \frac{\partial f_\alpha}{\partial x_i} dx_i$$

For $k>0$, this generalizes to

$$d\omega_\alpha = \sum_i \sum_{i_1 < \cdots < i_k} \frac{\partial \omega_\alpha^{i_1 \cdots i_k}}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where the right hand side is brought into the right form by the calculation rules $dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0$ if $i \in \{i_1, \ldots, i_k\}$ and $dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = (-1)^l dx_{i_1} \wedge \cdots \wedge dx_{i_l} \wedge dx_i \wedge dx_{i_{l+1}} \wedge \cdots \wedge dx_{i_k}$ if $i_1 < \cdots < i_l < i < i_{l+1} < \cdots < i_k$.

It is a not too difficult exercise to check that the local $(k+1)$-forms $d\omega_\alpha$ fit together to define a $(k+1)$-form on all of M. Two other exercises are: $d(d\omega) = 0$ and $d(f^*\omega) = f^*(d\omega)$ if $f:N \to M$ is a differentiable map.

The fundamental theorem of calculus now generalizes in the case of integrals over k-chains to the Stokes theorem

$$\int_c d\omega = \int_{\partial c} \omega$$

where c is a singular k-chain and ω is a $(k-1)$-form on M.

6.11. Manifolds with boundary

We defined a smooth manifold (without boundary) as a collection of subsets $U_\alpha \subset \mathbb{R}^n$ together with gluing data. This yields such things as the sphere surface. But not such things as the solid unit ball and the solid torus. These are manifolds with boundary which we now proceed to define. Let $H = \{x \in \mathbb{R}^n : x_n \geq 0\}$.

Now a manifold with boundary M is defined as a collection of open subsets U_α open in $\mathbb{R}^n$ or open in H with gluing data $\phi_{\alpha\beta}$: $U_{\alpha\beta} \to U_{\beta\alpha}$ as before which the additional requirement that

$$\phi_{\alpha\beta}(U_{\alpha\beta} \cap \partial H) = (U_{\beta\alpha} \cap \partial H)$$

where $\partial H = \{x \in \mathbb{R}^n : x_n = 0\}$, the boundary of H. (If $U_{\alpha\beta} \cap \partial H \neq \emptyset$ then differentiability of $\phi_{\alpha\beta}$ means (as always) that $\phi_{\alpha\beta}$ extends to a differentiable mapping on some subset open in $\mathbb{R}^n$ which contains $U_{\alpha\beta}$).

The $U_\alpha \cap \partial H$ and $U_{\alpha\beta} \cap \partial H$ are open subsets in $\mathbb{R}^{n-1}$ and the $\phi_{\alpha\beta}$ restricted to these subsets then define an $(n-1)$-manifold (without boundary) ∂M, the boundary of M. The tangent spaces (bundle) to M are again defined by means of the local pieces $U_\alpha \times \mathbb{R}^n$ (also for the points in $U_\alpha \cap \partial H$), and a Riemannian metric on M means again an inner product on all of the $T_x M$.

Let M be a Riemannian manifold with boundary ∂M. For $x \in \partial M$, $T_x \partial M$ is an $(n-1)$-dimensional subspace. Thus there are two vectors of unit length in $T_x M$ perpendicular to $T_x \partial M$. Precisely one of these points outwards (seen, as always, locally by going back to a $U_\alpha \subset H$). This defines the *outward normal* to ∂M at $x \in \partial M$.

An outward normal can also be defined in a slightly different setting. Let N be a oriented $(n-1)$-dimensional submanifold of an oriented Riemannian n-manifold M. For each $x \in M$ let $(v_1, ..., v_{n-1})$ be an orthonormal basis of $T_x N$ with the given orientation on N. Then there is precisely one unit length vector $v_n \in T_x M$ such that $(v_1, ..., v_{n-1}, v_n)$ is an orthonormal basis of $T_x M$ with the given orientation on M. In this setting v_n is also called the outward normal to N at $x \in N$.

6.12. The volume form.

We now know how to integrate k-forms over k-chains and in particular n-forms over n-manifolds. This still does not give meaning to, say, the integral over a sphere of a function on that sphere. For that we must find a good way of assigning n-forms to functions much like in ordinary one dimensional calculus one assigns the 1-form fdx to the function f.

The multidimensional analogue of this for manifolds is the volume form. Let V be a n-dimensional vectorspace with inner product and an orientation. An n-form on V is of the form $\omega = a\det$. For each orthonormal basis $(v_1, ..., v_1)$ we have $\omega(v_1, ..., v_n) = \pm a$. Thus there is precisely one n-form on V with the additional property that it takes the value 1 on each orthonormal basis with the given orientation. This one is called the volume element of V (determined by the inner product and the orientation).

Now let M be an oriented Riemannian manifold (with or without boundary). Then the volume form ω_M on M is defined by setting $\omega(x): T_x M \times \cdots \times T_x M \to \mathbb{R}$ equal to the unique volume element of $T_x M$ for each $x \in M$ (determined by the given inner product on $T_x M$ defined by the Riemannian metric and the given orientation).

More explicitly in terms of local coordinate patches U_α, the volume form can be described as follows. Let $\epsilon_\alpha = 1$ or -1 depending on whether the given orientations on $\{x\} \times \mathbb{R}^n$ agree with the standard orientation or not. For each x apply Gram-Schmidt orthonormalization with respect to the given inner product on $\{x\} \times \mathbb{R}^n$ to the standard basis $(e_1, ..., e_n)$, to obtain a differentiable family of orthonormal bases $\{v_1(x), ..., v_n(x)\}$. Now set

$$\omega_\alpha = \epsilon_\alpha \det(v_1(x), ..., v_n(x))^{-1} dx_1 \wedge \cdots \wedge dx_n$$

These ω_α are then the local pieces and gluing data description of the volume form ω_M often written as dV (even enough there may not be an $(n-1)$-form V such that $\omega_M = dV$).

Now of course if $(v_1,...,v_n)$ is the Gram-Schmidt orthonormalization of $(e_1,...,e_n)$, $v_i = Se_i$, then $S^TgS = I_n$ and hence $\det(S) = \det(g)^{\frac{1}{2}}$ so that the volume form is equal to

$$dV = \epsilon\det(g)^{\frac{1}{2}}dx_1\wedge\cdots\wedge dx_n$$

A function f on M is now integrated as

$$\int_M f = \int_M f\omega_M = \int_M f dV$$

6.13. Classical Stokes' type theorems.

A number of classical theorems now follow more or less directly from the general Stokes theorem 6.10.

Let $M\subset\mathbb{R}^2$ be a compact 2-dimensional manifold with boundary. E.g. a disk or an annulus. Let $f,g:M\to\mathbb{R}$ be differentiable. Then (Green's theorem)

$$\int_{\partial M} fdx+gdy = \int\int_M (\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y})dxdy$$

This results from the general Stokes' theorem of 6.10 (and the various remarks on defining integrals over manifolds instead of chains, cf 6.9), because

$$d(fdx+gdy) = \frac{\partial f}{\partial x}dx\wedge dx+\frac{\partial f}{\partial y}dy\wedge dx+\frac{\partial g}{\partial x}dx\wedge dy+\frac{\partial g}{\partial y}dy\wedge dy =$$
$$\frac{\partial g}{\partial x}dx\wedge dy-\frac{\partial f}{\partial y}dx\wedge dy.$$

For a vectorfield ψ on $\mathbb{R}^n$, $\psi = \sum\psi^i\frac{\partial}{\partial x_i}$ one defines the divergence by $\mathrm{div}(\psi) = \sum\frac{\partial\psi^i}{\partial x_i}$. The divergence theorem now says that for an oriented manifold with boundary $M\subset\mathbb{R}^3$ one has

$$\int_M \mathrm{div}(\psi)dV = \int_{\partial M} <\psi,n>dA$$

where dV is the volume form of the three dimensional manifold M, dA the volume form (area form) of the two-dimensional manifold ∂M. Here n is the outward normal to ∂M, and the inner products (i.e. the Riemannian structure) are induced from the standard ones on $\mathbb{R}^3$.

The curl of a vectorfield $\psi = \Sigma\psi^i\frac{\partial}{\partial x_i}$ on $\mathbb{R}^3$ is defined by

$$\mathrm{curl}(\psi) = (\frac{\partial\psi^3}{\partial x_2}-\frac{\partial\psi^2}{\partial x_3})\frac{\partial}{\partial x^1}+(\frac{\partial\psi^1}{\partial x_3}-\frac{\partial\psi^3}{\partial x_1})\frac{\partial}{\partial x_2}+(\frac{\partial\psi^1}{\partial x_2}-\frac{\partial\psi^2}{\partial x_1})\frac{\partial}{\partial x_3}$$

Let $M\subset\mathbb{R}^3$ be a compact, oriented, 2-dimensional manifold with boundary. Give ∂M an orientation such that together with the outward normal n its oriented bases give back the orientation of M. Let s parametrize ∂M and let ϕ be a vectorfield in ∂M such that $ds(\phi)=1$ (everywhere). Then the classical Stokes' formula says that

$$\int_M <\mathrm{curl}(\psi),n>dA = \int_{\partial M} <\psi,\phi>ds$$

All these theorems hold in greater generality. E.g. M could be a cube in the divergence theorem. To obtain those one uses either approximation arguments (smooth the corner and edges of the cube) or one can do the whole theory again with manifolds with corners and worse (which is possible).

All these theorems also generalize both to more general situations and to higher dimensions. To describe and discuss those, however, would bring in still more machinery such as the $*$-operator and contractions, though of course locally on the U_α it can all be done in terms of explicit formulas. For example the divergence of a vectorfield $\psi = \sum \psi^i \frac{\partial}{\partial x_i}$ (locally) on a Riemannian manifold is defined as the function

$$\operatorname{div}(\psi) = \sum_i \det(g)^{-\frac{1}{2}} \frac{\partial}{\partial x_i}(\det(g)^{\frac{1}{2}} \psi^i)$$

which of course fits the standard definition in the case of the Riemannian manifold $\mathbb{R}^n$ with $g = I_n$. One has $d(*\psi) = (dV)\operatorname{div}(\psi)$ and there results a higher dimensional divergence theorem.

7. Conclusion

The above is sort of a 'bare-bones-with-decorations' outline of manifolds and calculus on manifolds with a number of important omissions, notably contractions, the Poincaré lemma, the $*$-duality operator, connections and covariant differentiation, and curvature. It is at this point that things start getting interesting and it is at this point that this tutorial stops. Several lecture series in this volume will testify to the usefulness and power of all this machinery.